Available online at www.sciencedirect.com

SCIENCE DIRECT®

International Journal of Solids and Structures 43 (2006) 4720–4738

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

# Eshelby tensors for a spherical inclusion in microstretch elastic fields

A. Kiris [a], E. Inan [b,*]

[a] *Faculty of Science and Letters, Istanbul Technical University, 34469 Maslak, Istanbul, Turkey*
[b] *Faculty of Art and Sciences, Işık University, 34398 Maslak, Istanbul, Turkey*

Received 29 March 2005; received in revised form 14 June 2005
Available online 27 July 2005

**Abstract**

In the present work, microelastic and macroelastic fields are presented for the case of spherical inclusions embedded in an infinite microstretch material using the concept of Green's functions. The Eshelby tensors are obtained for a spherical inclusion and it is shown that their forms for microelongated, micropolar and the classical cases are the proper limiting cases of the Eshelby tensors of microstretch materials.


*Keywords:* Eshelby tensor; Eigenstrain; Microstretch; Microelongation; Micropolar; Green's functions

## 1. Introduction

A microcontinuum is considered as the collection of material particles, which can deform independently in the microscale in addition to the classical bulk deformation of the material. Eringen and Suhubi (1964) and Suhubi and Eringen (1964) introduced and developed a general theory for this phenomenon which is called micromorphic continua. As it is known, the general micromorphic theory is very complicated even for the linear case. To overcome the difficulties, Eringen introduced first the micropolar elasticity (Eringen, 1966) and, then the microstretch elasticity (Eringen, 1990). Because of their well suitability to the nature of many materials, both theories were universally accepted.

* Corresponding author. Tel.: +90 212 286 2962x2154; fax: +90 212 286 5796.
*E-mail address:* inan@isikun.edu.tr (E. Inan).

0020-7683/$ - see front matter 
doi:10.1016/j.ijsolstr.2005.06.028

Although the numbers of the unknown constitutive coefficients in micropolar and microstretch theories are considerably less than the general case, there are still some undetermined constitutive coefficients. Thus, we followed a different approach (Inan, 1990) and considered an additional homogenization procedure similar to the one known for the composite materials, to evaluate and estimate overall effective material properties. To apply the usual homogenization techniques (Mori–Tanaka method (Mori and Tanaka, 1973; Benveniste, 1987), for instance), we need to know the Eshelby tensors which establish the relation between the strains of the matrix material and of the inclusion. Several problems have been solved by using the Eshelby's equivalent inclusion theory (Eshelby, 1957). For the linear theory of the asymmetric elasticity, some solutions are given by Sandru (1966). Hsieh et al. (1980) and Hsieh (1982) have derived general formulas for the volume defects in micropolar media. Finally, Cheng and He (1995, 1997) obtained four Eshelby tensors for the spherical and the circular cylindrical inclusion in an isotropic centrosymmetric micropolar media, respectively. Sharma and Dasgupta (2002) calculated averaged stress and strains using numerical versions of the micropolar Eshelby tensors and extended the Mori–Tanaka method to the micropolar medium.

In the present work, the fundamental solutions are obtained for the microstretch medium. And then the Eshelby tensors are obtained for a spherical inclusion and it is shown that the Eshelby tensors for the microelongated, micropolar and the classical cases are the limiting cases of the Eshelby tensors of microstretch materials.

## 2. Fundamental solutions

The fundamental solutions for microelongated and micropolar media are given by Kiris and Inan (2005) and Cheng and He (1995), respectively. Applying a similar method given in Kiris and Inan (2005), we obtain the fundamental solutions and then the Eshelby tensors for microstretch medium.

As it is mentioned in the introduction, microstretch material is defined as the body with non-rigid particles which may do volume changes and microrotations in addition to the bulk deformation in the microstructural level. In other words, the material particles of such a material can stretch and contract independently of each others translations and rotations.

Since our task is to obtain the Eshelby tensors for the microstretch materials, first we shall obtain the field equations of the medium. The local forms of the equations of balance of momentum and moment of momentum at a point of a deformed microstretch body for the static case are given as (Eringen, 1999)

$$\begin{aligned} &t_{kl,k} + f_l = 0, \\ &m_{kl,k} + \in_{lmn} t_{mn} + l_l = 0, \\ &m_{k,k} + t - s + l = 0, \end{aligned} \tag{1}$$

where $t_{kl}$ is the stress tensor, $s_{kl}$ and $m_{kl}$ are the couple stress tensors, $m_k$ is the microstretch vector and $t = t_{kk}$, $s = s_{kk}$. $f_l$, $l_l$ and $l$ are the body force, the body moment and the body force densities, respectively. Subscripts preceded by a comma stand for derivatives with respect to the corresponding spatial coordinates and $\in_{lmn}$ is the permutation symbol.

The geometrical definitions and relations are given below:

$$\varepsilon_{kl} = u_{l,k} + \in_{lkm} \phi_m, \quad \gamma_{kl} = \phi_{k,l}, \quad \gamma_k = 3\theta_{,k}, \quad e = 3\theta. \tag{2}$$

Here, $\varepsilon_{kl}$ is the strain tensor, $\phi_k$ is the microrotation vector, $\theta$ is the microelongation and $u_k$ is the displacement vector.

The linearized constitutive equations for a microstretch medium are given by Eringen (1999) as

$$
\begin{aligned}
t_{kl} &= A^s_{kl}\theta + A^s_{mkl}\theta_{,m} + A_{klmn}\varepsilon_{mn} + C_{klmn}\gamma_{mn},\\
m_{kl} &= B^s_{lk}\theta + B^s_{\mathrm{mlk}}\theta_{,m} + C_{mnlk}\varepsilon_{mn} + B_{lkmn}\gamma_{mn},\\
m_k &= C^s_k\theta + C^s_{kl}\theta_{,l} + A^s_{klm}\varepsilon_{lm} + B^s_{klm}\gamma_{lm},\\
s - t &= C^s\theta + C^s_k\theta_{,k} + A^s_{kl}\varepsilon_{kl} + B^s_{kl}\gamma_{kl},
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
&A^s_{ij} = \lambda_0\delta_{ij}, \quad A^s_{kij} = 0, \quad A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \left(\mu + \frac{\chi}{2}\right)\delta_{ik}\delta_{jl} + \left(\mu - \frac{\chi}{2}\right)\delta_{il}\delta_{jk},\\
&B^s_{ij} = 0, \quad B^s_{kij} = b_0\in_{kij}, \quad B_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{il}\delta_{jk} + \gamma\delta_{ik}\delta_{jl}\\
&C^s = \lambda_1, \quad C^s_i = 0, \quad C^s_{ij} = a_0\delta_{ij}, \quad C_{ijkl} = 0.
\end{aligned}
\tag{4}
$$

Here $\lambda$, $\mu$ are Lamé constants, $\alpha$, $\beta$, $\gamma$ and $\chi$ are new constitutive coefficients due to the micropolar character of the medium, $a_0$, $\lambda_0$ and $\lambda_1$ are some new constitutive coefficients due to the microelongation and they are given as (Eringen, 1999)

$$
\begin{aligned}
a_0 &= 6\tau_1 + 6\tau_2 + 9\tau_3 + \tau_4 + 2\tau_5 + \tau_6 + 3\tau_7 + 2\tau_8 + \tau_9 + 3\tau_{10} + \tau_{11},\\
\lambda_0 &= 3\upsilon + 2\sigma - \chi,\\
\lambda_1 &= 9\tau + 6\eta + 3\chi.
\end{aligned}
\tag{5}
$$

We substitute the constitutive equations into the balance equations to obtain the fields equations for the microstretch medium. Thus, we find

$$
\begin{aligned}
&\left(\mu + \frac{\chi}{2}\right)u_{l,kk} + \left(\lambda + \mu - \frac{\chi}{2}\right)u_{k,kl} + \lambda_0\theta_{,l} + \chi\in_{lkm}\phi_{m,k} + f_l = 0,\\
&\gamma\phi_{l,kk} + (\alpha + \beta)\phi_{k,kl} + \chi\in_{lkm}u_{m,k} - 2\chi\phi_l + l_l = 0,\\
&a_0\theta_{,kk} - \lambda_1\theta - \lambda_0 u_{k,k} + l = 0.
\end{aligned}
\tag{6}
$$

For the Galerkin's Representation (Galerkin, 1930), it is more convenient to write all seven of Eq. (6) in the matrix form as

$$
\mathbf{M}\begin{bmatrix} u_1\\ u_2\\ u_3\\ \phi_1\\ \phi_2\\ \phi_3\\ \theta \end{bmatrix} = -\begin{bmatrix} f_1\\ f_2\\ f_3\\ l_1\\ l_2\\ l_3\\ l \end{bmatrix}.
\tag{7}
$$

To achieve this, we define first,

$$
\begin{aligned}
&X_i = \frac{\partial}{\partial x_i}, \quad \varDelta = X_1^2 + X_2^2 + X_3^2, \quad \Diamond_1 = (\lambda + 2\mu)\varDelta, \quad \Diamond_2 = \left(\mu + \frac{\chi}{2}\right)\varDelta,\\
&\Diamond_3 = a_0\varDelta - \lambda_1, \quad \Diamond_4 = \gamma\varDelta - 2\chi, \quad \Diamond_5 = (\alpha + \beta + \gamma)\varDelta - 2\chi
\end{aligned}
\tag{8}
$$

and then find $\mathbf{M}$ as

$$\mathbf{M}=\begin{bmatrix} \Diamond_2+(\lambda+\mu-\frac{\chi}{2})X_1^2 & (\lambda+\mu-\frac{\chi}{2})X_1X_2 & (\lambda+\mu-\frac{\chi}{2})X_1X_3 & 0 & -\chi X_3 & \chi X_2 & \lambda_0X_1 \\ (\lambda+\mu-\frac{\chi}{2})X_1X_2 & \Diamond_2+(\lambda+\mu-\frac{\chi}{2})X_2^2 & (\lambda+\mu-\frac{\chi}{2})X_2X_3 & \chi X_3 & 0 & -\chi X_1 & \lambda_0X_2 \\ (\lambda+\mu-\frac{\chi}{2})X_1X_3 & (\lambda+\mu-\frac{\chi}{2})X_2X_3 & \Diamond_2+(\lambda+\mu-\frac{\chi}{2})X_3^2 & -\chi X_2 & \chi X_1 & 0 & \lambda_0X_3 \\ 0 & -\chi X_3 & \chi X_2 & \Diamond_4+(\alpha+\beta)X_1^2 & (\alpha+\beta)X_1X_2 & (\alpha+\beta)X_1X_3 & 0 \\ \chi X_3 & 0 & -\chi X_1 & (\alpha+\beta)X_1X_2 & \Diamond_4+(\alpha+\beta)X_2^2 & (\alpha+\beta)X_2X_3 & 0 \\ -\chi X_2 & \chi X_1 & 0 & (\alpha+\beta)X_1X_3 & (\alpha+\beta)X_2X_3 & \Diamond_4+(\alpha+\beta)X_3^2 & 0 \\ -\lambda_0X_1 & -\lambda_0X_2 & -\lambda_0X_3 & 0 & 0 & 0 & \Diamond_3 \end{bmatrix}. \tag{9}$$

Now, denoting the inverse matrix of $\mathbf{M}$ by

$$\mathbf{M}^{-1}=\frac{(n_{ij})}{\Box_3\Box_4\Box_7}, \tag{10}$$

where

$$\begin{aligned} &n_{ii}=\Box_7\left(\Box_1-\Box_2X_i^2\right), \quad i=1,2,3,\\ &n_{ii}=\Box_4\left(\Box_5-\Box_6X_{i-3}^2\right), \quad i=4,5,6,\\ &n_{ii}=\Box_3\Box_7\Box_8, \quad i=7,\\ &n_{ij}=n_{ji}=-\Box_2\Box_7X_iX_j, \quad i\neq j, \quad i,j=1,2,3,\\ &n_{ij}=n_{ji}=-\Box_4\Box_6X_{i-3}X_{j-3}, \quad i\neq j,\ \ i,j=4,5,6,\\ &n_{7i}=-n_{i7}=\Box_3\Box_7\lambda_0X_i, \quad i=1,2,3,\\ &n_{14}=n_{25}=n_{36}=n_{41}=n_{52}=n_{63}=n_{7i}=n_{i7}=0,\ \ i=4,5,6,\\ &n_{15}=-n_{24}=n_{42}=-n_{51}=\Box_4\Box_7\chi X_3,\\ &n_{16}=-n_{34}=n_{43}=-n_{61}=-\Box_4\Box_7\chi X_2,\\ &n_{26}=-n_{35}=n_{53}=-n_{62}=\Box_4\Box_7\chi X_1 \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\Box_1=\Diamond_1\Diamond_3\Diamond_4+\lambda_0^2\Diamond_4\Delta,\\ &\Box_2=\left[\lambda_0^2+\left(\lambda+\mu-\frac{\chi}{2}\right)\Diamond_3\right]\Diamond_4-\chi^2\Diamond_3,\\ &\Box_3=\Diamond_2\Diamond_4+\chi^2\Delta,\\ &\Box_4=\Diamond_1\Diamond_3+\lambda_0^2\Delta,\\ &\Box_5=\Diamond_2\Diamond_5,\\ &\Box_6=\left[(\alpha+\beta)\Diamond_2-\chi^2\right],\\ &\Box_7=\Diamond_5,\\ &\Box_8=\Diamond_1. \end{aligned} \tag{12}$$

The solution of the matrix equation (7) is written as

$$\begin{bmatrix} u_1\\ u_2\\ u_3\\ \phi_1\\ \phi_2\\ \phi_3\\ \theta \end{bmatrix}=\mathbf{N}\begin{bmatrix} F_1\\ F_2\\ F_3\\ L_1\\ L_2\\ L_3\\ L \end{bmatrix}. \tag{13}$$

Here

$$\mathbf{N} = (n_{ij})_{7\times 7} \tag{14}$$

and

$$\Box_3\Box_4\Box_7\mathbf{F} = -\mathbf{f}, \quad \Box_3\Box_4\Box_7\mathbf{L} = -\mathbf{l}, \quad \Box_3\Box_4\Box_7 L = -l \tag{15}$$

From Eqs. (13) and (14) we find

$$\begin{aligned}
\mathbf{u} &= \Diamond_4(\Diamond_1\Diamond_3 + \lambda_0^2\Delta)\boldsymbol{\varphi} - \left\{\lambda_0^2\Diamond_4 + \left[\left(\lambda + \mu - \frac{\chi}{2}\right)\Diamond_4 - \chi^2\right]\Diamond_3\right\}\nabla\nabla\cdot\boldsymbol{\varphi} - \chi\Diamond_5\nabla\times\boldsymbol{\varphi}^* - \lambda_0\Diamond_5\nabla\varphi^{**},\\
\boldsymbol{\phi} &= \Diamond_2\Diamond_5\boldsymbol{\varphi}^* - \left[(\alpha+\beta)\Diamond_2 - \chi^2\right]\nabla\nabla\cdot\boldsymbol{\varphi}^* - \chi(\Diamond_1\Diamond_3 + \lambda_0^2\Delta)\nabla\times\boldsymbol{\varphi},\\
\theta &= \lambda_0(\Diamond_2\Diamond_4 + \chi^2\Delta)\nabla\cdot\boldsymbol{\varphi} + \Diamond_1\Diamond_5\varphi^{**}.
\end{aligned} \tag{16}$$

Here

$$\boldsymbol{\varphi} = \Diamond_5\mathbf{F}, \quad \boldsymbol{\varphi}^* = [\Diamond_1\Diamond_3 + \lambda_0^2\Delta]\mathbf{L}, \quad \varphi^{**} = [\Diamond_2\Diamond_4 + \chi^2\Delta]L \tag{17}$$

and equations for them become

$$\begin{aligned}
&(\Diamond_1\Diamond_3 + \lambda_0^2\Delta)(\Diamond_2\Diamond_4 + \chi^2\Delta)\boldsymbol{\varphi} = -\mathbf{f},\\
&\Diamond_5(\Diamond_2\Diamond_4 + \chi^2\Delta)\boldsymbol{\varphi}^* = -\mathbf{l},\\
&\Diamond_5(\Diamond_1\Diamond_3 + \lambda_0^2\Delta)\varphi^{**} = -l.
\end{aligned} \tag{18}$$

Substituting the open forms of the operators given by Eq. (8) into Eqs. (16) and (18), we arrive at,

$$\begin{aligned}
\mathbf{u} &= (\gamma\Delta - 2\chi)[(\lambda+2\mu)\Delta(a_0\Delta - \lambda_1) + \lambda_0^2\Delta]\boldsymbol{\varphi}\\
&\quad - \left\{\lambda_0^2(\gamma\Delta - 2\chi) + \left[\left(\lambda + \mu - \frac{\chi}{2}\right)\gamma\Delta - 2\chi(\lambda+\mu)\right](a_0\Delta - \lambda_1)\right\}\nabla\nabla\cdot\boldsymbol{\varphi}\\
&\quad - \chi[(\alpha+\beta+\gamma)\Delta - 2\chi]\nabla\times\boldsymbol{\varphi}^* - \lambda_0[(\alpha+\beta+\gamma)\Delta - 2\chi]\nabla\varphi^{**},\\
\boldsymbol{\phi} &= \left(\mu + \frac{\chi}{2}\right)\Delta[(\alpha+\beta+\gamma)\Delta - 2\chi]\boldsymbol{\varphi}^* - \left[(\alpha+\beta)\left(\mu + \frac{\chi}{2}\right)\Delta - \chi^2\right]\nabla\nabla\cdot\boldsymbol{\varphi}^*\\
&\quad - \chi\left[(\lambda+2\mu)\Delta(a_0\Delta - \lambda_1) + \lambda_0^2\Delta\right]\nabla\times\boldsymbol{\varphi},\\
\theta &= \lambda_0\Delta\left[\left(\mu + \frac{\chi}{2}\right)\gamma\Delta - 2\mu\chi\right]\nabla\cdot\boldsymbol{\varphi} + (\lambda+2\mu)\Delta[(\alpha+\beta+\gamma)\Delta - 2\chi]\varphi^{**}
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&[(\lambda+2\mu)(a_0\Delta - \lambda_1) + \lambda_0^2]\Delta^2\left[\left(\mu + \frac{\chi}{2}\right)\gamma\Delta - 2\mu\chi\right]\boldsymbol{\varphi} = -\mathbf{f},\\
&[(\alpha+\beta+\gamma)\Delta - 2\chi]\left[\left(\mu + \frac{\chi}{2}\right)\gamma\Delta^2 - 2\mu\chi\Delta\right]\boldsymbol{\varphi}^* = -\mathbf{l},\\
&[(\alpha+\beta+\gamma)\Delta - 2\chi][(\lambda+2\mu)\Delta(a_0\Delta - \lambda_1) + \lambda_0^2\Delta]\varphi^{**} = -l.
\end{aligned} \tag{20}$$

In the first step of our formulation we assume $\mathbf{l} = \mathbf{0}$, $l = 0$ and body force field $\mathbf{f}$ as irrotational. Thus, we write

$$\mathbf{f} = \nabla\pi_0. \tag{21}$$

Then from Eqs. (16) and (18), we find

$$(\Diamond_1\Diamond_3 + \lambda_0^2\Delta)\Lambda_0 = -\pi_0. \tag{22}$$

Here

$$(\Diamond_2\Diamond_4 + \chi^2\Delta)\boldsymbol{\varphi} = \nabla\Lambda_0. \tag{23}$$

Now, solutions for this case are obtained simply as

$$\begin{aligned}&\mathbf{u} = \Diamond_3 \nabla \Lambda_0,\\&\boldsymbol{\phi} = \mathbf{0},\\&\theta = \lambda_0 \nabla \cdot (\nabla \Lambda_0).\end{aligned} \tag{24}$$

For the solenoidal force field, we have $\boldsymbol{\pi}$, such that

$$\mathbf{f} = \nabla \times \boldsymbol{\pi}. \tag{25}$$

In the same way, from Eqs. (16) and (18) we obtain

$$(\Diamond_2 \Diamond_4 + \chi^2 \Delta)\boldsymbol{\Lambda} = -\boldsymbol{\pi}. \tag{26}$$

Here

$$(\Diamond_1 \Diamond_3 + \lambda_0^2 \Delta)\boldsymbol{\varphi} = \nabla \times \boldsymbol{\Lambda}. \tag{27}$$

Then the solutions become

$$\begin{aligned}&\mathbf{u} = \nabla \times (\Diamond_4 \boldsymbol{\Lambda}),\\&\boldsymbol{\phi} = -\chi \nabla \times (\nabla \times \boldsymbol{\Lambda}),\\&\theta = 0.\end{aligned} \tag{28}$$

In the second step, we assume $\mathbf{f} = \mathbf{0}$, $l = 0$ and the body moments, $\mathbf{l}$ are nonzero. Considering an irrotational field first, we write

$$\mathbf{l} = \nabla \pi_0^*. \tag{29}$$

Then we find

$$\Diamond_3 \Lambda_0^* = -\pi_0^*. \tag{30}$$

Here

$$(\Diamond_2 \Diamond_4 + \chi^2 \Delta)\boldsymbol{\varphi}^* = \nabla \Lambda_0^* \tag{31}$$

and the solutions are

$$\begin{aligned}&\mathbf{u} = \mathbf{0},\\&\boldsymbol{\phi} = \nabla \Lambda_0^*,\\&\theta = 0.\end{aligned} \tag{32}$$

For solenoidal moment field, we take

$$\mathbf{l} = \nabla \times \boldsymbol{\pi}^* \tag{33}$$

and obtain

$$(\Diamond_2 \Diamond_4 + \chi^2 \Delta)\boldsymbol{\Lambda}^* = -\boldsymbol{\pi}^*. \tag{34}$$

Here

$$\Diamond_3 \boldsymbol{\varphi}^* = \nabla \times \boldsymbol{\Lambda}^* \tag{35}$$

and the solutions are

$$\begin{aligned} &\mathbf{u} = -\chi\nabla \times (\nabla \times \mathbf{\Lambda}^*),\\ &\boldsymbol{\phi} = \nabla \times (\Diamond_2 \mathbf{\Lambda}^*),\\ &\theta = 0. \end{aligned} \tag{36}$$

As the last step, we assume $\mathbf{f} = \mathbf{0}$, $\mathbf{l} = \mathbf{0}$ and $l$ is nonzero. This time we define

$$\Diamond_3 \varphi^{**} = \Lambda_0^{**}. \tag{37}$$

Then, from Eqs. (16) and (18), we have

$$(\Diamond_1 \Diamond_3 + \lambda_0^2 \Delta)\Lambda_0^{**} = -l. \tag{38}$$

Thus, the solutions are

$$\begin{aligned} &\mathbf{u} = -\lambda_0 \nabla \Lambda_0^{**},\\ &\boldsymbol{\phi} = \mathbf{0},\\ &\theta = \Diamond_1 \Lambda_0^{**}. \end{aligned} \tag{39}$$

In the following part of this section, we will determine the fundamental solutions for an infinite medium for a force field $\mathbf{q}$, concentrated at the origin of the coordinate system. Using Helmholtz decomposition, Eqs. (22) and (26) are written as

$$\begin{aligned} &\left[(\lambda + 2\mu)a_0 \Delta^2 - ((\lambda + 2\mu)\lambda_1 - \lambda_0^2)\Delta\right]\Lambda_0 = \frac{1}{4\pi}\mathbf{q} \cdot \nabla\left(\frac{1}{r}\right),\\ &\left[\left(\mu + \frac{\chi}{2}\right)\gamma \Delta^2 - 2\mu\chi\Delta\right]\mathbf{\Lambda} = \frac{1}{4\pi}\mathbf{q} \times \nabla\left(\frac{1}{r}\right), \end{aligned} \tag{40}$$

where $r = |\mathbf{x}|$. The solutions are given as

$$\begin{aligned} &\Lambda_0 = -\frac{1}{8\pi B_0}\left(\frac{\mathbf{q} \cdot \mathbf{r}}{r}\right) - \frac{h_1^2}{4\pi B_0}\mathbf{q} \cdot \nabla\left(\frac{1}{r}(1 - \mathrm{e}^{-r/h_1})\right),\\ &\mathbf{\Lambda} = \frac{1}{8\pi B_3}\nabla \times (\mathbf{q}r) + \frac{h_2^2}{4\pi B_3}\nabla \times \left(\frac{\mathbf{q}}{r}(1 - \mathrm{e}^{-r/h_2})\right), \end{aligned} \tag{41}$$

where

$$\begin{aligned} &B_0 = (\lambda + 2\mu)\lambda_1 - \lambda_0^2, \quad B_2 = \left(\mu + \frac{\chi}{2}\right)\gamma,\\ &B_1 = (\lambda + 2\mu)a_0, \quad B_3 = 2\mu\chi,\\ &h_1^2 = \frac{B_1}{B_0}, \quad h_2^2 = \frac{B_2}{B_3}. \end{aligned} \tag{42}$$

On the other hand, from Eqs. (24) and (28), we write

$$\begin{aligned} &\mathbf{u} = (a_0\Delta - \lambda_1)\nabla\Lambda_0 + (\gamma\Delta - 2\chi)\nabla \times \mathbf{\Lambda},\\ &\boldsymbol{\phi} = -\chi\nabla \times (\nabla \times \mathbf{\Lambda}),\\ &\theta = \lambda_0 \nabla \cdot (\nabla\Lambda_0). \end{aligned} \tag{43}$$

Now, substituting the results of Eq. (41) into Eq. (43), we find the displacement, microrotation and micro-elongation as

$$\begin{aligned}
\mathbf{u} &= \frac{(\lambda+3\mu)\lambda_1-\lambda_0^2}{8\pi\mu B_0}\left(\frac{\mathbf{q}}{r}\right)+\frac{a_0}{4\pi B_0}\left(\frac{\mathbf{q}}{r^3}\right)+\frac{(\lambda+\mu)\lambda_1-\lambda_0^2}{8\pi\mu B_0}\left(\frac{(\mathbf{q}\cdot\mathbf{r})\mathbf{r}}{r^3}\right)-\frac{3a_0}{4\pi B_0}\left(\frac{(\mathbf{q}\cdot\mathbf{r})\mathbf{r}}{r^5}\right)\\
&\quad -\frac{\lambda_0^2}{4\pi(\lambda+2\mu)B_0}\left(\frac{\mathbf{q}}{r}\mathrm{e}^{-r/h_1}\right)+\frac{a_0}{4\pi B_0}\nabla\times\nabla\times\left(\frac{\mathbf{q}}{r}\mathrm{e}^{-r/h_1}\right)+\frac{\lambda_1 B_1}{4\pi B_0^2}\nabla\times\nabla\times\left(\frac{\mathbf{q}}{r}(1-\mathrm{e}^{-r/h_1})\right)\\
&\quad -\frac{\gamma}{16\pi\mu^2}\nabla\times\nabla\times\left(\frac{\mathbf{q}}{r}(1-\mathrm{e}^{-r/h_2})\right),\\
\boldsymbol{\phi} &= \frac{1}{8\pi\mu}\nabla\times\left(\frac{\mathbf{q}}{r}(1-\mathrm{e}^{-r/h_2})\right),\\
\theta &= \frac{\lambda_0}{4\pi B_0}\left(\frac{\mathbf{q}\cdot\mathbf{r}}{r^3}\right)+\frac{\lambda_0}{4\pi B_0}\nabla\cdot\left(\frac{\mathbf{q}}{r}\mathrm{e}^{-r/h_1}\right).
\end{aligned} \tag{44}$$

In the sequel, we will obtain the solutions of the next two steps. Thus, we assume that concentrated body moment $\mathbf{p}$ is acting at the origin of the coordinate frame. This case is represented by Eqs. (30) and (34). Then we write

$$\begin{aligned}
&[(\alpha+\beta+\gamma)\Delta-2\chi]\Lambda_0^* = \frac{1}{4\pi}\mathbf{p}\cdot\nabla\left(\frac{1}{r}\right),\\
&\left[\left(\mu+\frac{\chi}{2}\right)\gamma\Delta^2-2\mu\chi\Delta\right]\boldsymbol{\Lambda}^* = \frac{1}{4\pi}\mathbf{p}\times\nabla\left(\frac{1}{r}\right).
\end{aligned} \tag{45}$$

The solutions of these equations are

$$\begin{aligned}
\Lambda_0^* &= -\frac{1}{4\pi B_4}\mathbf{p}\cdot\nabla\left(\frac{1}{r}(1-\mathrm{e}^{-r/h_3})\right),\\
\boldsymbol{\Lambda}^* &= \frac{1}{8\pi B_3}\nabla\times(\mathbf{p}r)+\frac{h_2^2}{4\pi B_3}\nabla\times\left(\frac{\mathbf{p}}{r}(1-\mathrm{e}^{-r/h_2})\right).
\end{aligned} \tag{46}$$

Here

$$B_4 = 2\chi,\quad B_5 = \alpha+\beta+\gamma,\quad h_3^2 = B_5/B_4. \tag{47}$$

On the other hand by combining Eqs. (32) and (36), we may write

$$\begin{aligned}
\mathbf{u} &= -\chi\nabla\times(\nabla\times\boldsymbol{\Lambda}^*),\\
\boldsymbol{\phi} &= \nabla\Lambda_0^*+\left(\mu+\frac{\chi}{2}\right)\Delta\nabla\times\boldsymbol{\Lambda}^*,\\
\theta &= 0.
\end{aligned} \tag{48}$$

Now substituting the solutions of (46) into Eq. (48) for the second step, we obtain the displacement, microrotation and the microelongation as

$$\begin{aligned}
\mathbf{u} &= \frac{1}{8\pi\mu}\nabla\times\left(\frac{\mathbf{p}}{r}(1-\mathrm{e}^{-r/h_2})\right),\\
\boldsymbol{\phi} &= \frac{1}{4\pi B_5}\left(\frac{\mathbf{p}}{r}\mathrm{e}^{-r/h_3}\right)-\frac{1}{4\pi B_4}\nabla\times\nabla\times\left(\frac{\mathbf{p}}{r}(1-\mathrm{e}^{-r/h_3})\right)+\frac{(2\mu+\chi)}{8\pi B_3}\nabla\times\nabla\times\left(\frac{\mathbf{p}}{r}(1-\mathrm{e}^{-r/h_2})\right),\\
\theta &= 0.
\end{aligned} \tag{49}$$

Finally we write $\nabla l = \bar{\mathbf{l}}$ and $\bar{\mathbf{l}} = \bar{\mathbf{p}}\delta(x_1, x_2, x_3)$ for the concentrated force density at the origin of the coordinate system. Then from Eq. (38), we find

$$\{(\lambda + 2\mu)a_0\Delta^2 - [(\lambda + 2\mu)\lambda_1 - \lambda_0^2]\Delta\}\Lambda_0^{**} = \frac{1}{4\pi}\bar{\mathbf{p}} \cdot \nabla\left(\frac{1}{r}\right). \tag{50}$$

Solution of this equation is

$$\Lambda_0^{**} = -\frac{1}{8\pi B_0}\left(\frac{\bar{\mathbf{p}} \cdot \mathbf{r}}{r}\right) - \frac{h_1^2}{4\pi B_0}\bar{\mathbf{p}} \cdot \nabla\left(\frac{1}{r}(1 - \mathrm{e}^{-r/h_1})\right). \tag{51}$$

In the same way, from Eq. (39), we write

$$\begin{aligned} &\mathbf{u} = -\lambda_0 \nabla \Lambda_0^{**}, \\ &\boldsymbol{\phi} = \mathbf{0}, \\ &\theta = (\lambda + 2\mu)\Delta\Lambda_0^{**}. \end{aligned} \tag{52}$$

Finally, substituting the expression (51) into Eq. (52), we find the solutions corresponding to the third step;

$$\begin{aligned} &\mathbf{u} = \frac{\lambda_0}{8\pi B_0}\left[\frac{\bar{\mathbf{p}}}{r} - \frac{(\bar{\mathbf{p}} \cdot \mathbf{r})\bar{\mathbf{r}}}{r^3}\right] - \frac{\lambda_0}{4\pi B_0}\left(\frac{\bar{\mathbf{p}}}{r}\mathrm{e}^{-r/h_1}\right) + \frac{\lambda_0 h_1^2}{4\pi B_0}\nabla \times \nabla \times \left(\frac{\bar{\mathbf{p}}}{r}(1 - \mathrm{e}^{-r/h_1})\right), \\ &\boldsymbol{\phi} = \mathbf{0}, \\ &\theta = -\frac{(\lambda + 2\mu)}{4\pi B_0}\nabla \cdot \left(\frac{\bar{\mathbf{p}}}{r}(1 - \mathrm{e}^{-r/h_1})\right). \end{aligned} \tag{53}$$

Brief summaries for the similar problems in microelongated and micropolar media which will be used for comparison in the later part of the work are given in Appendix A.

## 3. Eshelby tensors of microstretch medium

In this section, Eshelby tensors for the microstretch medium will be obtained. As it is known, one of the major problems in Mori–Tanaka method is the determination of Eshelby tensors which establish the relations between the deformations of the matrix material and of the inclusions. These tensors are obtained by Cheng and He (1995, 1997) for micropolar medium with spherical and cylindrical inclusions respectively and by Kiris and Inan (2005) for microelongated medium with spherical inclusions. The results of these works are summarized in Appendix B.

In the classical theory of elasticity, Mura (1982) defined the concept of "eigenstrain" as a nonelastic deformation which occurs as an additional deformation to the elastic deformations, and the concept of "eigenstress" as the stress due to these eigenstrains. In the similar fashion, Hsieh et al. (1980) and Cheng and He (1995, 1997) introduced the concepts of "stress-free microstrain" and "eigentorsion", respectively and finally Inan (1990) introduced "microeigenstrain" concept in the microstructural level. To describe the deformations in an infinite microstretch material with inclusions, eigenstrains and microeigenstrains will be defined as follows:

$$\varepsilon_{ij}^t = \varepsilon_{ij}^*\Lambda(\Omega), \quad \gamma_{ij}^t = \gamma_{ij}^*\Lambda(\Omega), \quad \theta^t = \theta^*\Lambda(\Omega), \tag{54}$$

$$\Lambda(\Omega) = \begin{cases} 1, & \mathbf{x} \in \Omega, \\ 0, & \mathbf{x} \in \mathbb{R}^3 - \Omega. \end{cases} \tag{55}$$

Here, $\Omega$ is a subdomain of the infinite body occupied by the inclusion and the quantities with superscript "$*$" and "$t$" denote the eigenstrains of the inclusion and general eigenstrains, respectively.

The geometric relations of the microstretch medium are given by Eq. (2). Then the constitutive equations due to an inclusion take the following form:

$$\begin{aligned}
t_{ij} &= A^s_{ij}(\theta-\theta^t) + A^s_{kij}(\theta_{,k}-\theta^t_{,k}) + A_{ijkl}(\varepsilon_{kl}-\varepsilon^t_{kl}) + C_{ijkl}(\gamma_{kl}-\gamma^t_{kl}),\\
m_{ij} &= B^s_{ji}(\theta-\theta^t) + B^s_{kji}(\theta_{,k}-\theta^t_{,k}) + C_{klji}(\varepsilon_{kl}-\varepsilon^t_{kl}) + B_{jikl}(\gamma_{kl}-\gamma^t_{kl}),\\
m_i &= C^s_i(\theta-\theta^t) + C^s_{ij}(\theta_{,j}-\theta^t_{,j}) + A^s_{ijk}(\varepsilon_{jk}-\varepsilon^t_{jk}) + B^s_{ijk}(\gamma_{jk}-\gamma^t_{jk}),\\
s-t &= C^s(\theta-\theta^t) + C^s_i(\theta_{,i}-\theta^t_{,i}) + A^s_{ij}(\varepsilon_{ij}-\varepsilon^t_{ij}) + B^s_{ij}(\gamma_{ij}-\gamma^t_{ij}).
\end{aligned} \tag{56}$$

Here, the constitutive coefficients of the linear isotropic microstretch medium are given by Eq. (4).

Now, we may obtain the final form of the fundamental equations by substituting constitutive equations (56), the geometric relations (2) and the constitutive coefficients (4) into the equilibrium equations (1). Then we get

$$\begin{aligned}
&\lambda_0\theta_{,i} + \left(\lambda+\mu-\frac{\chi}{2}\right)u_{j,ij} + \left(\mu+\frac{\chi}{2}\right)u_{i,jj} + \chi\in_{ijk}\phi_{k,j} + f_i + f^t_i = 0,\\
&(\alpha+\beta)\phi_{j,ij} + \gamma\phi_{i,jj} + \chi\in_{ijk}u_{k,j} - 2\chi\phi_i + l_i + l^t_i = 0,\\
&a_0\theta_{,ii} - \lambda_1\theta - \lambda_0 u_{i,i} + l + l^t = 0.
\end{aligned} \tag{57}$$

The terms that have the superscript "$t$" in above equations are easily determined by the use of the balance Eq. (1). Thus, we find

$$\begin{aligned}
&f^t_i = -t^t_{ji,j}, \quad l^t_i = -m^t_{ji,j} - \in_{ijk}t^t_{jk}, \quad l^t = -m^t_{i,i} - t^t + s^t,\\
&t^t_{ij} = A^s_{ij}\theta^t + A_{ijkl}\varepsilon^t_{kl}, \quad m^t_{ij} = B^s_{kji}\theta^t_{,k} + B_{jikl}\gamma^t_{kl},\\
&m^t_i = C^s_{ij}\theta^t_{,j} + B^s_{ijk}\gamma^t_{jk}, \quad s^t - t^t = C^s\theta^t + A^s_{ij}\varepsilon^t_{ij}.
\end{aligned} \tag{58}$$

The unknown quantities $\mathbf{u}$, $\boldsymbol{\phi}$ and $\theta$ in Eq. (57) may be determined by the use of the Green's function approach. As it is known, only one Green's function is sufficient to find the solution of the corresponding problem in the classical theory of elasticity and four Green's functions for each microelongated and micropolar media. Thus, we need total of nine Green's function for the microstretch medium to determine all the unknowns. To obtain the equations for the first set of Green's functions, we assume that only the body force $\mathbf{f}$ is acting to the origin of the coordinate system while the body moment $\mathbf{l}$ and body force density $l$ are absent. Then we have

$$\begin{aligned}
&\lambda_0 g_{n,i} + \left(\lambda+\mu-\frac{\chi}{2}\right)\mathcal{G}_{jn,ij} + \left(\mu+\frac{\chi}{2}\right)\mathcal{G}_{in,jj} + \chi\in_{ijk}G_{kn,j} + \delta_{in}\delta(\mathbf{x}-\mathbf{x}') = 0,\\
&(\alpha+\beta)G_{jn,ij} + \gamma G_{in,jj} + \chi\in_{ijk}\mathcal{G}_{kn,j} - 2\chi G_{in} = 0,\\
&a_0 g_{n,ii} - \lambda_1 g_n - \lambda_0\mathcal{G}_{in,i} = 0.
\end{aligned} \tag{59}$$

To obtain the next set of the equations, this time, we assume only the body moment $\mathbf{l}$ is acting at the origin of the coordinate system while the body force $\mathbf{f}$ and body force density $l$ are absent. Then we find

$$\begin{aligned}
&\lambda_0\hat{g}_{n,i} + \left(\lambda+\mu-\frac{\chi}{2}\right)\widehat{\mathcal{G}}_{jn,ij} + \left(\mu+\frac{\chi}{2}\right)\widehat{\mathcal{G}}_{in,jj} + \chi\in_{ijk}\widehat{G}_{kn,j} = 0,\\
&(\alpha+\beta)\widehat{G}_{jn,ij} + \gamma\widehat{G}_{in,jj} + \chi\in_{ijk}\widehat{\mathcal{G}}_{kn,j} - 2\chi\widehat{G}_{in} + \delta_{in}\delta(\mathbf{x}-\mathbf{x}') = 0,\\
&a_0\hat{g}_{n,ii} - \lambda_1\hat{g}_n - \lambda_0\widehat{\mathcal{G}}_{in,i} = 0.
\end{aligned} \tag{60}$$

For the last set of the equations, we consider that, only the body force density $l$ is acting to the origin of the coordinate system while the body force $\mathbf{f}$ and body moment $\mathbf{l}$ are absent. Thus, we arrive at,

$$\begin{aligned}
&\lambda_0 \hat{\hat{g}}_{n,i} + \left(\lambda + \mu - \frac{\chi}{2}\right) \widehat{\widehat{\mathscr{G}}}_{jn,ij} + \left(\mu + \frac{\chi}{2}\right) \widehat{\widehat{\mathscr{G}}}_{in,jj} + \chi \in_{ijk} \widehat{\widehat{\mathscr{G}}}_{kn,j} = 0, \\
&(\alpha + \beta) \widehat{\widehat{G}}_{jn,ij} + \gamma \widehat{\widehat{G}}_{in,jj} + \chi \in_{ijk} \widehat{\widehat{\mathscr{G}}}_{kn,j} - 2\chi \widehat{\widehat{\mathscr{G}}}_{in} = 0, \\
&a_0 \hat{\hat{g}}_{n,ii} - \lambda_1 \hat{\hat{g}}_n - \lambda_0 \widehat{\widehat{\mathscr{G}}}_{in,i} + \delta_{in} I_i \delta(\mathbf{x} - \mathbf{x}') = 0.
\end{aligned} \tag{61}$$

Here, $\mathscr{G}_{in}$, $\widehat{\mathscr{G}}_{in}$ and $\widehat{\widehat{\mathscr{G}}}_{in}$ denote the Green's functions corresponding to $u_i$, $G_{in}$, $\hat{G}_{in}$ and $\hat{\hat{G}}_{in}$ for $\phi_i$, and $g_n$, $\hat{g}_n$ and $\hat{\hat{g}}_n$ for $\theta$ due to the three different loading types mentioned above and $I_i = -(1/4\pi)(1/r)_{,i}$.

Using the solution method given in Section 2, Green's functions of the general case are found as follows:

$$\begin{aligned}
&\mathscr{G}_{kn}(\mathbf{x} - \mathbf{x}') = \mathscr{G}^{\mathrm{C}}_{kn}(\mathbf{x} - \mathbf{x}') + \mathscr{G}^{\mathrm{P}}_{kn}(\mathbf{x} - \mathbf{x}') + G^{\mathrm{E}}_{kn}(\mathbf{x} - \mathbf{x}'), \\
&\widehat{\mathscr{G}}_{kn}(\mathbf{x} - \mathbf{x}') = G_{kn}(\mathbf{x} - \mathbf{x}') = \frac{1}{8\pi\mu} \in_{knl} \left(\frac{\mathrm{e}^{-r/h_2} - 1}{r}\right)_{,l}, \\
&\widehat{\widehat{\mathscr{G}}}_{kn}(\mathbf{x} - \mathbf{x}') = \frac{\lambda_0 B_1}{4\pi B_0^2} \left[\frac{1}{h_1^2} \frac{r_{,kn}}{2} + \left(\frac{1 - \mathrm{e}^{-r/h_1}}{r}\right)_{,kn}\right], \\
&\widehat{G}_{kn}(\mathbf{x} - \mathbf{x}') = -\frac{1}{16\pi\mu} \left(\frac{\mathrm{e}^{-r/h_2} - 1}{r}\right)_{,kn} + \frac{1}{8\pi\chi} \left(\frac{\mathrm{e}^{-r/h_3} - \mathrm{e}^{-r/h_2}}{r}\right)_{,kn} + \frac{2\mu + \chi}{16\pi\mu\chi h_2^2} \frac{\mathrm{e}^{-r/h_2}}{r} \delta_{kn}, \\
&\widehat{\widehat{G}}_{kn}(\mathbf{x} - \mathbf{x}') = 0, \\
&g_n(\mathbf{x} - \mathbf{x}') = -\frac{\lambda_0}{4\pi B_0} \left(\frac{1 - \mathrm{e}^{-r/h_1}}{r}\right)_{,n}, \\
&\hat{g}_n(\mathbf{x} - \mathbf{x}') = 0, \\
&\hat{\hat{g}}_n(\mathbf{x} - \mathbf{x}') = -\frac{\lambda + 2\mu}{4\pi B_0} \left(\frac{1 - \mathrm{e}^{-r/h_1}}{r}\right)_{,n}.
\end{aligned} \tag{62}$$

Here

$$\begin{aligned}
&\mathscr{G}^{\mathrm{C}}_{kn}(\mathbf{x} - \mathbf{x}') = \frac{1}{8\pi\mu} \left(2 \frac{\delta_{kn}}{r} - \frac{\lambda + \mu}{\lambda + 2\mu} r_{,kn}\right), \\
&\mathscr{G}^{\mathrm{P}}_{kn}(\mathbf{x} - \mathbf{x}') = \frac{B}{4\pi} \left[h_2^2 \left(\frac{\mathrm{e}^{-r/h_2} - 1}{r}\right)_{,kn} - \delta_{kn} \frac{\mathrm{e}^{-r/h_2}}{r}\right], \\
&\mathscr{G}^{\mathrm{E}}_{kn}(\mathbf{x} - \mathbf{x}') = \frac{a_0 \lambda_0^2}{4\pi B_0^2} \left[\frac{1}{h_1^2} \frac{r_{,kn}}{2} + \left(\frac{1 - \mathrm{e}^{-r/h_1}}{r}\right)_{,kn}\right],
\end{aligned} \tag{63}$$

where $B = \chi/[\mu(2\mu + \chi)]$ and the superscripts "C, P, E" denoted the classical, micropolar and microelongation quantities. As it is clearly seen from the first expression of Eq. (62), the Green's function $\mathscr{G}_{kn}$ corresponding to the displacement vector $u_k$ due to the application of body force, $\mathbf{f}$ is the sum of the three Green's functions corresponding to classical, micropolar and microelongation cases. Besides, the Green's functions $\hat{g}_n$ corresponding to microelongation $\theta$ due to the application of the body moments and the Green's functions $\widehat{\widehat{G}}_{kn}$ corresponding to microrotation $\phi_k$ due to the application of body force density are obtained as zero as expected.

Now, to derive the solutions for $u_k$, $\phi_k$ and $\theta$ satisfying Eq. (57) in terms of the solutions of the Green's functions, we employ the reciprocity theorem

$$\int_V (\overline{F}_k u'_k - \overline{F}'_k u_k) \,\mathrm{d}V + \int_V (\overline{C}_k \phi'_k - \overline{C}'_k \phi_k) \,\mathrm{d}V + \int_V (\overline{L}\theta' - \overline{L}'\theta) \,\mathrm{d}V = 0. \tag{64}$$

Here

$$\overline{F}_k = f_k^t + f_k, \quad \overline{C}_k = l_k^t + l_k, \quad \overline{L} = l^t + l. \tag{65}$$

For the present problem, we write

$$\begin{aligned}\{u_k', \phi_k', \theta', \overline{F}_k', \overline{C}_k', \overline{L}'\} &= \{\mathscr{G}_{kn}, G_{kn}, g_n, \delta_{kn}\delta(\mathbf{x}-\mathbf{x}'), 0, 0\} = \left\{\widehat{\mathscr{G}}_{kn}, \widehat{G}_{kn}, \hat{g}_n = 0, 0, \delta_{kn}\delta(\mathbf{x}-\mathbf{x}'), 0\right\} \\ &= \left\{\widehat{\widehat{\mathscr{G}}}_{kn}, \widehat{\widehat{G}}_{kn} = 0, \hat{\hat{g}}_n, 0, 0, 0, \delta_{kn} I_k \delta(\mathbf{x}-\mathbf{x}')\right\}\end{aligned} \tag{66}$$

in (64) and express the three field quantities as

$$\begin{aligned}u_n(\mathbf{x}) &= \int_V [\overline{F}_k(\mathbf{x}')\mathscr{G}_{kn}(\mathbf{x}-\mathbf{x}') + \overline{C}_k(\mathbf{x}')G_{kn}(\mathbf{x}-\mathbf{x}') + \overline{L}(\mathbf{x}')g_n(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}\mathbf{x}', \\ \phi_n(\mathbf{x}) &= \int_V [\overline{F}_k(\mathbf{x}')\widehat{\mathscr{G}}_{kn}(\mathbf{x}-\mathbf{x}') + \overline{C}_k(\mathbf{x}')\hat{G}_{kn}(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}\mathbf{x}', \\ \theta(\mathbf{x}) &= \int_V [(\overline{F}_k(\mathbf{x}')\widehat{\widehat{\mathscr{G}}}_{kn}(\mathbf{x}-\mathbf{x}') + \overline{L}(\mathbf{x}')\hat{\hat{g}}_n(\mathbf{x}-\mathbf{x}'))/I_n(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}\mathbf{x}'.\end{aligned} \tag{67}$$

Here, the quantities $f_i^t$, $l_i^t$ and $l^t$ in Eq. (58) may be regarded as the fictitious body forces, body moments and body force density. Now, substituting the definitions given by Eqs. (58) and (65) into Eq. (67) and integrating by parts with the assumption of vanishing boundary terms, we obtain

$$\begin{aligned}u_n(\mathbf{x}) &= -\int_V [\lambda_0\theta^t\mathscr{G}_{kn,k} + A_{klij}\varepsilon_{ij}^t\mathscr{G}_{\ln,k} + B_{klij}\gamma_{ij}^t G_{kn,l} - \chi\in_{jik}\varepsilon_{ij}^t G_{kn} + a_0\theta^t g_{n,kk} - \lambda_1\theta^t g_n - \lambda_0\delta_{ij}\varepsilon_{ij}^t g_n]\,\mathrm{d}\mathbf{x}', \\ \phi_n(\mathbf{x}) &= -\int_V [\lambda_0\theta^t\widehat{\mathscr{G}}_{kn,k} + A_{klij}\varepsilon_{ij}^t\widehat{\mathscr{G}}_{\ln,k} + B_{klij}\gamma_{ij}^t\widehat{G}_{kn,l} - \chi\in_{jik}\varepsilon_{ij}^t\widehat{G}_{kn}]\,\mathrm{d}\mathbf{x}', \\ \theta(\mathbf{x}) &= -\int_V [(\lambda_0\theta^t\widehat{\widehat{\mathscr{G}}}_{kn,k} + A_{klij}\varepsilon_{ij}^t\widehat{\widehat{\mathscr{G}}}_{\ln,k} + a_0\theta^t\hat{\hat{g}}_{n,kk} - \lambda_1\theta^t\hat{\hat{g}}_n - \lambda_0\delta_{ij}\varepsilon_{ij}^t\hat{\hat{g}}_n)/I_n]\,\mathrm{d}\mathbf{x}'.\end{aligned} \tag{68}$$

Eq. (68) express the displacement vector $u_k$, the microrotation vector $\phi_k$, and the microelongation scalar $\theta$ in terms of the Green's functions. Now, strain, microrotation, microelongation, stress, couple stress and other microquantities of a microstretch medium may be easily found by the use of the results given in Eqs. (68), (3) and (2), respectively.

### 3.1. Elastic field due to an inclusion in a microstretch material

In this section, we consider an inclusion occupying a subdomain $\Omega$ in an infinite microelongated medium. Now assuming the asymmetric eigenstrain $\varepsilon_{ij}^*$, eigentorsion $\gamma_{ij}^*$, and the microeigenstrain $\theta^*$ in Eq. (54) are constants over the inclusion (Cheng and He, 1995, 1997), we write Eq. (68) as in the following form:

$$\begin{aligned}u_n(\mathbf{x}) &= u_n^{\mathrm{C}}(\mathbf{x}) + I_{nij}(\mathbf{x})\varepsilon_{ij}^* + J_{nij}(\mathbf{x})\gamma_{ij}^* + K_n(\mathbf{x})\theta^*, \\ \phi_n(\mathbf{x}) &= \widehat{I}_{nij}(\mathbf{x})\varepsilon_{ij}^* + \widehat{J}_{nij}(\mathbf{x})\gamma_{ij}^*, \\ \theta(\mathbf{x}) &= \widehat{\widehat{I}}_{ij}(\mathbf{x})\varepsilon_{ij}^* + \widehat{\widehat{K}}(\mathbf{x})\theta^*.\end{aligned} \tag{69}$$

Here, the coefficients of eigenstrain, eigentorsion and microeigenstrain are

$$
\begin{aligned}
I_{nij}(\mathbf{x}) &= -\frac{\chi}{2}(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})\int_\Omega \mathscr{G}^{\mathrm{C}}_{kn,l}\,\mathrm{d}\mathbf{x}' - A_{lkij}\int_\Omega(\mathscr{G}_{kn,l}-\mathscr{G}^{\mathrm{C}}_{kn,l})\,\mathrm{d}\mathbf{x}' + \chi\in_{jik}\int_\Omega G_{kn}\,\mathrm{d}\mathbf{x}' + \lambda_0\delta_{ij}\int_\Omega g_n\,\mathrm{d}\mathbf{x}',\\
J_{nij}(\mathbf{x}) &= -B_{klij}\int_\Omega G_{kn,l}\,\mathrm{d}\mathbf{x}',\\
K_n(\mathbf{x}) &= -\lambda_0\int_\Omega \mathscr{G}_{kn,k}\,\mathrm{d}\mathbf{x}' - a_0\int_\Omega g_{n,kk}\,\mathrm{d}\mathbf{x}' + \lambda_1\int_\Omega g_n\,\mathrm{d}\mathbf{x}',\\
\widehat{I}_{nij}(\mathbf{x}) &= -A_{lkij}\int_\Omega \widehat{\mathscr{G}}_{kn,l}\,\mathrm{d}\mathbf{x}' + \chi\in_{jik}\int_\Omega \widehat{G}_{kn}\,\mathrm{d}\mathbf{x}',\\
\widehat{J}_{nij}(\mathbf{x}) &= -B_{klij}\int_\Omega \widehat{G}_{kn,l}\,\mathrm{d}\mathbf{x}',\\
\widehat{\widehat{I}}_{ij}(\mathbf{x}) &= -A_{lkij}\int_\Omega \frac{\widehat{\widehat{\mathscr{G}}}_{kn,l}}{I_n}\,\mathrm{d}\mathbf{x}' + \lambda_0\delta_{ij}\int_\Omega \frac{\hat{\hat{g}}_n}{I_n}\,\mathrm{d}\mathbf{x}',\\
\widehat{\widehat{K}}(\mathbf{x}) &= -\lambda_0\int_\Omega \frac{\widehat{\widehat{\mathscr{G}}}_{kn,k}}{I_n}\,\mathrm{d}\mathbf{x}' - a_0\int_\Omega \frac{\hat{\hat{g}}_{n,kk}}{I_n}\,\mathrm{d}\mathbf{x}' + \lambda_1\int_\Omega \frac{\hat{\hat{g}}_n}{I_n}\,\mathrm{d}\mathbf{x}'
\end{aligned}
\tag{70}
$$

and

$$
\begin{aligned}
u^{\mathrm{C}}_n(\mathbf{x}) &= I^{\mathrm{C}}_{nij}(\mathbf{x})\varepsilon^*_{ij},\\
I^{\mathrm{C}}_{nij}(\mathbf{x}) &= -(\lambda\delta_{ij}\delta_{kl}+\mu\delta_{ik}\delta_{jl}+\mu\delta_{il}\delta_{jk})\int_\Omega \mathscr{G}^{\mathrm{C}}_{kn,l}\,\mathrm{d}\mathbf{x}'.
\end{aligned}
\tag{71}
$$

Using Eqs. (62) and (63) in (70) and (71) and after some mathematical manipulations, we arrive

$$
\begin{aligned}
I_{nij}(\mathbf{x}) &= I^{\mathrm{P}}_{nij}(\mathbf{x}) + I^{\mathrm{E}}_{nij}(\mathbf{x}),\\
J_{nij}(\mathbf{x}) &= -\frac{1}{2\mu}\left[\gamma\in_{nik}M_{1,jk}(\mathbf{x}) + \beta\in_{njk}M_{1,ik}(\mathbf{x})\right] + \frac{1}{2\mu}\left[\gamma\in_{nik}M_{3,jk}(\mathbf{x},h_2) + \beta\in_{njk}M_{3,ik}(\mathbf{x},h_2)\right],\\
K_n(\mathbf{x}) &= -2\frac{\lambda_0\lambda_1}{B_0}M_{1,n}(\mathbf{x}) + \frac{a_0\lambda_0\left(2\lambda_0^2-(\lambda+2\mu)\lambda_1\right)}{B_0^2}M_{3,kkn}(\mathbf{x},h_1) + \frac{\lambda_1\lambda_0}{B_0}M_{3,n}(\mathbf{x},h_1),\\
\hat{I}_{nij}(\mathbf{x}) &= \frac{1}{4\mu}\left[\chi\in_{jik}M_{1,kn}(\mathbf{x}) - (2\mu+\chi)\in_{njk}M_{1,ik}(\mathbf{x}) - (2\mu-\chi)\in_{nik}M_{1,jk}(\mathbf{x})\right]\\
&\quad - \frac{1}{4\mu}\left[(2\mu+\chi)\in_{jik}M_{3,kn}(\mathbf{x},h_2) - (2\mu+\chi)\in_{njk}M_{3,ik}(\mathbf{x},h_2) - (2\mu-\chi)\in_{nik}M_{3,jk}(\mathbf{x},h_2)\right]\\
&\quad + \frac{1}{2}\in_{jik}M_{3,kn}(\mathbf{x},h_3) + \frac{2\mu+\chi}{4\mu h_2^2}\in_{jin}M_3(\mathbf{x},h_2),\\
\hat{J}_{nij}(\mathbf{x}) &= -\frac{\gamma+\beta}{4\mu}M_{1,ijn}(\mathbf{x}) + \frac{2\mu+\chi}{4\mu\chi}\left[\alpha\delta_{ij}M_{3,kkn}(\mathbf{x},h_2) + (\gamma+\beta)M_{3,ijn}(\mathbf{x},h_2)\right]\\
&\quad - \frac{1}{2\chi}\left[\alpha\delta_{ij}M_{3,kkn}(\mathbf{x},h_3) + (\gamma+\beta)M_{3,ijn}(\mathbf{x},h_3)\right]\\
&\quad - \frac{2\mu+\chi}{4\mu\chi h_2^2}\left[\alpha\delta_{ij}M_{3,n}(\mathbf{x},h_2) + \gamma\delta_{in}M_{3,j}(\mathbf{x},h_2) + \beta\delta_{jn}M_{3,i}(\mathbf{x},h_2)\right],
\end{aligned}
$$

$$
\begin{aligned}
\hat{\hat{I}}_{ij}(\mathbf{x}) &= \frac{2\lambda_0(\lambda+\mu)}{B_0}\delta_{ij} + \frac{\lambda_0\mu}{B_0}\frac{M_{2,ijn}(\mathbf{x})}{M_{1,n}(\mathbf{x})} + \frac{2\lambda_0 B_1\mu}{B_0^2}\frac{M_{1,ijn}(\mathbf{x})}{M_{1,n}(\mathbf{x})} - \frac{\lambda\lambda_0 B_1}{B_0^2}\delta_{ij}\frac{M_{3,kkn}(\mathbf{x},h_1)}{M_{1,n}(\mathbf{x})} \\
&\quad - \frac{2\lambda_0 B_1\mu}{B_0^2}\frac{M_{3,ijn}(\mathbf{x},h_1)}{M_{1,n}(\mathbf{x})} - \frac{\lambda_0(\lambda+2\mu)}{B_0}\delta_{ij}\frac{M_{3,n}(\mathbf{x},h_1)}{M_{1,n}(\mathbf{x})}, \\
\hat{\hat{K}}(\mathbf{x}) &= \frac{\lambda_0^2+(\lambda+2\mu)\lambda_1}{B_0} - \frac{B_1\left[2\lambda_0^2-(\lambda+2\mu)\lambda_1\right]}{B_0^2}\frac{M_{3,kkn}(\mathbf{x},h_1)}{M_{1,n}(\mathbf{x})} - \frac{(\lambda+2\mu)\lambda_1}{B_0}\frac{M_{3,n}(\mathbf{x},h_1)}{M_{1,n}(\mathbf{x})},
\end{aligned} \tag{72}
$$

where

$$
\begin{aligned}
I^{\mathrm{C}}_{nij}(\mathbf{x}) &= \frac{\lambda+\mu}{\lambda+2\mu}M_{2,ijn}(\mathbf{x}) - \frac{\lambda}{\lambda+2\mu}\delta_{ij}M_{1,n}(\mathbf{x}) - \delta_{in}M_{1,j}(\mathbf{x}) - \delta_{jn}M_{1,i}(\mathbf{x}), \\
I^{\mathrm{P}}_{nij}(\mathbf{x}) &= 2\mu B h_2^2 M_{1,ijn}(\mathbf{x}) + \frac{\chi}{\mu}\left(\delta_{in}M_{1,j}(\mathbf{x}) - \delta_{jn}M_{1,i}(\mathbf{x})\right) \\
&\quad - B h_2^2\left(\lambda\delta_{ij}M_{3,kkn}(\mathbf{x},h_2) + 2\mu M_{3,ijn}(\mathbf{x},h_2)\right) + B\lambda\delta_{ij}M_{3,n}(\mathbf{x},h_2) \\
&\quad + \left[B\left(\mu+\frac{\chi}{2}\right)+\frac{\chi}{2\mu}\right]\delta_{jn}M_{3,i}(\mathbf{x},h_2) + \left[B\left(\mu-\frac{\chi}{2}\right)-\frac{\chi}{2\mu}\right]\delta_{in}M_{3,j}(\mathbf{x},h_2), \\
I^{\mathrm{E}}_{nij}(\mathbf{x}) &= -\frac{2(\lambda+\mu)\lambda_0^2}{(\lambda+2\mu)B_0}\delta_{ij}M_{1,n}(\mathbf{x}) - \frac{\lambda_0^2}{(\lambda+2\mu)B_0}\mu M_{2,ijn}(\mathbf{x}) + \frac{a_0\lambda_0^2}{B_0^2}\lambda\delta_{ij}M_{3,kkn}(\mathbf{x},h_1) \\
&\quad + \frac{2a_0\lambda_0^2}{B_0^2}\mu M_{3,ijn}(\mathbf{x},h_1) - \frac{2a_0\lambda_0^2}{B_0^2}\mu M_{1,ijn}(\mathbf{x}) + \frac{\lambda_0^2}{B_0}\delta_{ij}M_{3,n}(\mathbf{x},h_1),
\end{aligned} \tag{73}
$$

here

$$
M_{3,kkn}(\mathbf{x},h) = \frac{1}{h^2}M_{3,n}(\mathbf{x},h) \tag{74}
$$

and we define the following potential functions

$$
M_1(\mathbf{x}) = \frac{1}{4\pi}\int_\Omega \frac{1}{r}\mathrm{d}\mathbf{x}', \quad M_2(\mathbf{x}) = \frac{1}{4\pi}\int_\Omega r\,\mathrm{d}\mathbf{x}', \quad M_3(\mathbf{x},h) = \frac{1}{4\pi}\int_\Omega \frac{\mathrm{e}^{-r/h}}{r}\mathrm{d}\mathbf{x}'. \tag{75}
$$

As it is mentioned above, Eshelby tensor for an isotropic elastic body in classical elasticity are found by two integrals which are the same of the first two integrals of Eq. (75) and they are given explicitly by Mura (1982). Therefore, the problem of determining Eshelby tensors for a microstretch solid is basically converted to the determination of third integral given by Eq. (75). The results for a spherical inclusion with radius $a$ is given in (Cheng and He, 1995) as:

$$
\begin{aligned}
M_1(\mathbf{x}) &= \frac{1}{4\pi}\int_\Omega \frac{1}{r}\mathrm{d}\mathbf{x}' = \begin{cases} -\dfrac{1}{6}(x^2-3a^2), & \mathbf{x}\in\Omega, \\ \dfrac{a^3}{3x}, & \mathbf{x}\in\mathbb{R}^3-\Omega, \end{cases} \\
M_2(\mathbf{x}) &= \frac{1}{4\pi}\int_\Omega r\,\mathrm{d}\mathbf{x}' = \begin{cases} -\dfrac{1}{60}(x^4-10a^2x^2-15a^4), & \mathbf{x}\in\Omega, \\ \dfrac{a^3}{15}\left(5x+\dfrac{a^2}{x}\right), & \mathbf{x}\in\mathbb{R}^3-\Omega, \end{cases} \\
M_3(\mathbf{x},h) &= \frac{1}{4\pi}\int_\Omega \frac{\mathrm{e}^{-r/h}}{r}\mathrm{d}\mathbf{x}' = \begin{cases} h^2 - h^2(h+a)\dfrac{\sinh x/h}{x}\mathrm{e}^{-a/h}, & \mathbf{x}\in\Omega, \\ h^2\left(a\cosh\dfrac{a}{h} - h\sinh\dfrac{a}{h}\right)\dfrac{\mathrm{e}^{-x/h}}{x}, & \mathbf{x}\in\mathbb{R}^3-\Omega. \end{cases}
\end{aligned} \tag{76}
$$

Here, $x = |\mathbf{x}|$. Using Eqs. (2), (69) and (71), we express the strain, microtorsion and microelongation in a microstretch material as

$$\begin{aligned}
\varepsilon_{kl}(\mathbf{x}) &= K_{klij}(\mathbf{x})\varepsilon^*_{ij} + L_{klij}(\mathbf{x})\gamma^*_{ij} + M_{kl}(\mathbf{x})\theta^*,\\
\gamma_{kl}(\mathbf{x}) &= \hat{K}_{klij}(\mathbf{x})\varepsilon^*_{ij} + \hat{L}_{klij}(\mathbf{x})\gamma^*_{ij},\\
\theta(\mathbf{x}) &= \hat{\hat{K}}_{ij}(\mathbf{x})\varepsilon^*_{ij} + \hat{\hat{M}}(\mathbf{x})\theta^*.
\end{aligned} \tag{77}$$

Here,

$$\begin{aligned}
&K_{klij}(\mathbf{x}) = I^{\mathrm{C}}_{lij,k}(\mathbf{x}) + I_{lij,k}(\mathbf{x}) - \in_{klm}\hat{I}_{mij}(\mathbf{x}), \quad L_{klij}(\mathbf{x}) = J_{lij,k}(\mathbf{x}) - \in_{klm}\hat{J}_{mij}(\mathbf{x})\\
&M_{kl}(\mathbf{x}) = K_{l,k}(\mathbf{x}), \quad \hat{K}_{klij}(\mathbf{x}) = \hat{I}_{kij,l}(\mathbf{x}), \quad \hat{L}_{klij}(\mathbf{x}) = \hat{J}_{kij,l}(\mathbf{x}),\\
&\hat{\hat{K}}_{ij}(\mathbf{x}) = \hat{\hat{I}}_{ij}(\mathbf{x}), \quad \hat{\hat{M}}(\mathbf{x}) = \hat{\hat{K}}(\mathbf{x}).
\end{aligned} \tag{78}$$

These tensors are the modified version of the classical Eshelby tensors for microstretch materials including spherical inclusions. They are not homogeneous over the inclusion even for spherical case, unlike the classical theory of elasticity.

We may obtain the solutions due to microelongation, micropolar and classical cases as the special cases of the microstretch theory. To arrive the solutions of microelongation, the micropolar constitutive coefficients are assumed absent. In the same way, to find the solutions for micropolar theory, the constitutive coefficients due to microelongation are taken zero. To get the result for the classical theory both quantities, due to microelongation and micropolar cases are assumed absent. That is,

$$\phi_k = \theta = 0, \quad \chi = \alpha = \beta = \gamma = a_0 = \lambda_0 = \lambda_1 = 0 \tag{79}$$

and we have only

$$K_{klij}(\mathbf{x}) = I^{\mathrm{C}}_{lij,k}(\mathbf{x}) \tag{80}$$

and all the other Eshelby tensors are zero.

Eshelby tensor $S_{ijkl}$ in classical theory of elasticity is defined as:

$$\varepsilon_{ij} = S_{ijkl}\varepsilon^*_{kl}, \tag{81}$$

to arrive this result as a limit case of the present problem, we write from Eq. (77)$_1$

$$\frac{1}{2}(\varepsilon_{kl} + \varepsilon_{lk}) = \frac{1}{2}\left[I^{\mathrm{C}}_{lij,k}(\mathbf{x}) + I^{\mathrm{C}}_{kij,l}(\mathbf{x})\right]\frac{1}{2}\left(\varepsilon^*_{ij} + \varepsilon^*_{ji}\right). \tag{82}$$

Here $I^{\mathrm{C}}_{lij,k}(\mathbf{x})$ is symmetric with respect to the indices $i$ and $j$. On the other hand $\varepsilon_{kl}$ is a symmetric tensor in classical elasticity. Thus, the comparison of Eqs. (81) and (82) gives

$$S_{klij} = \frac{1}{2}\left[I^{\mathrm{C}}_{lji,k}(\mathbf{x}) + I^{\mathrm{C}}_{kji,l}(\mathbf{x})\right]. \tag{83}$$

For a spherical inclusion, Eq. (83) takes the following form:

$$S_{klij} = \frac{5v-1}{15(1-v)}\delta_{ij}\delta_{kl} + \frac{4-5v}{15(1-v)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{84}$$

This result is the same of the well-known Eshelby tensor for a spherical inclusion in the classical theory of elasticity (Mura, 1982).

## 4. Conclusions

In this work, we have obtained the Eshelby tensors for isotropic homogeneous microstretch materials with a spherical inclusion. It is also shown that the solutions corresponding to microelongation, micropolar and classical cases are all the special cases of the microstretch theory. Using the obtained Eshelby tensors, the

Mori–Tanaka method can be extended to the microstretch medium which we shall give the details in a further work and determine the overall material moduli of damaged materials modeled as a microstretch continuum.

## Appendix A

### A.1. Solutions for microelongated medium

The general equations for the microelongated elastic fields and the fundamental equations are obtained by Kiris and Inan (2005). Then the Eshelby tensors are found for the spherical inclusion in microelongated medium (Kiris and Inan, 2005).

As it is known, microelongation theory is defined as a special case of microstretch theory which ignores the effects of microrotations. The local balance of momentum and moment of momentum at a point of a deformed microelongated medium are (Kiris and Inan, 2005)

$$\begin{aligned} &t_{kl,k}+f_l=0,\\ &m_{k,k}+t-s+l=0. \end{aligned} \tag{A.1}$$

Geometrical relations and the linearized constitutive equations are summarized as in the following:

$$\varepsilon_{kl}=u_{l,k},\quad \gamma_k=3\theta_{,k}, \tag{A.2}$$

$$\begin{aligned} &t_{kl}=A^s_{kl}\theta+A^s_{mkl}\theta_{,m}+A_{klmn}\varepsilon_{mn},\\ &s-t=C^s\theta+C^s_k\theta_{,k}+A^s_{kl}\varepsilon_{kl},\\ &m_k=C^s_k\theta+C^s_{kl}\theta_{,l}+A^s_{klm}\varepsilon_{lm}. \end{aligned} \tag{A.3}$$

Here again $f_k$ is the body force, $l$ is the microelongation force density, $u_k$ is the displacement vector, $\theta$ is the microelongation, $t_{kl}$ and $s_{kl}$ are stress tensors, $m_k$ is microelongation vector and $t=t_{kk}$, $s=s_{kk}$.

Constitutive coefficients of (A.3) for the linearized isotropic microelongated medium are

$$\begin{aligned} &A^s_{kl}=\lambda_0\delta_{kl},\quad A^s_{klm}=0,\quad C^s=\lambda_1,\quad C^s_k=0,\quad C^s_{kl}=a_0\delta_{kl},\\ &A_{klmn}=\lambda\delta_{kl}\delta_{mn}+\mu\delta_{km}\delta_{\ln}+\mu\delta_{kn}\delta_{lm},\quad C_{klmn}=0. \end{aligned} \tag{A.4}$$

To obtain the governing equations for microelongated medium we substitute Eqs. (A.2)–(A.4) into Eq. (A.1). The result is

$$\begin{aligned} &\lambda_0\theta_{,l}+(\lambda+\mu)u_{k,kl}+\mu\, u_{l,kk}+f_l=0,\\ &a_0\theta_{,kk}-\lambda_1\theta-\lambda_0u_{k,k}+l=0. \end{aligned} \tag{A.5}$$

Here $\lambda$ and $\mu$ are Lamé constants, $a_0$, $\lambda_0$ and$\lambda_1$ are new constitutive coefficients due to microelongation and they are given in Eq. (5).

The results of the Galerkin's representation for microelongated medium are given by Kiris and Inan (2005) and the fundamental solution for infinite, elastic microelongated medium are

$$\begin{aligned} \mathbf{u}=&\frac{B_0+\lambda_1\mu}{8\pi\mu B_0}\left(\frac{\mathbf{q}}{r}\right)+\frac{B_0-\lambda_1\mu}{8\pi\mu B_0}\left(\frac{(\mathbf{q}\cdot\mathbf{r})\mathbf{r}}{r^3}\right)+\frac{a_0}{4\pi B_0}\left(\frac{\mathbf{q}}{r^3}\right)-\frac{3a_0}{4\pi B_0}\left(\frac{(\mathbf{q}\cdot\mathbf{r})\mathbf{r}}{r^5}\right)\\ &+\frac{a_0-\lambda_1h_1^2}{4\pi B_0}\nabla\times\nabla\times\left(\frac{\mathbf{q}}{r}\mathrm{e}^{-r/h_1}\right)+\frac{a_0-\lambda_1h_1^2}{4\pi B_0h_1^2}\left(\frac{\mathbf{q}}{r}\mathrm{e}^{-r/h_1}\right)+\frac{\lambda_1h_1^2}{4\pi B_0}\nabla\times\nabla\times\left(\frac{\mathbf{q}}{r}\right),\\ \theta=&\frac{\lambda_0}{4\pi B_0}\left(\frac{\mathbf{q}\cdot\mathbf{r}}{r^3}\right)+\frac{\lambda_0}{4\pi B_0}\nabla\cdot\left(\frac{\mathbf{q}}{r}\mathrm{e}^{-r/h_1}\right). \end{aligned} \tag{A.6}$$

Here, $\mathbf{q}$ is the concentrated force acting at the origin of the coordinate system.

Solutions for the second case which we assume $\nabla l = \bar{\mathbf{l}}$ and $\bar{\mathbf{l}} = \bar{\mathbf{p}}\delta(x_1, x_2, x_3)$ acting at the origin of the coordinate system are found as (Kiris and Inan, 2005)

$$\begin{aligned} \mathbf{u} &= \frac{\lambda_0}{8\pi B_0}\left[\frac{\bar{\mathbf{p}}}{r} - \frac{(\bar{\mathbf{p}}\cdot\mathbf{r})\bar{\mathbf{r}}}{r^3}\right] - \frac{\lambda_0}{4\pi B_0}\left(\frac{\bar{\mathbf{p}}}{r}\mathrm{e}^{-r/h_1}\right) + \frac{\lambda_0 h_1^2}{4\pi, B_0}\nabla\times\nabla\times\left(\frac{\bar{\mathbf{p}}}{r}(1-\mathrm{e}^{-r/h_1})\right), \\ \theta &= -\frac{(\lambda+2\mu)}{4\pi B_0}\nabla\cdot\left(\frac{\bar{\mathbf{p}}}{r}(1-\mathrm{e}^{-r/h_1})\right). \end{aligned} \tag{A.7}$$

### A.2. Solutions for micropolar medium

As it is known, micropolar medium is defined by Eringen with the assumption that the material particles are rigid and can only rotate independently in addition to the bulk deformation (Eringen, 1999). General equations of micropolar medium are given by Eringen (1999) and the fundamental solutions are obtained by Sandru (1966). Then Cheng and He (1995) found the Eshelby tensors for a spherical inclusion. The results of these two papers are given in the followings.

The equations of equilibrium for micropolar medium (Eringen, 1999):

$$\begin{aligned} &t_{kl,k} + f_l = 0, \\ &m_{kl,k} + \in_{lmn} t_{mn} + l_l = 0, \end{aligned} \tag{A.8}$$

Geometrical relations:

$$\varepsilon_{kl} = u_{l,k} + \in_{lkm}\phi_m, \quad \gamma_{kl} = \phi_{k,l} \tag{A.9}$$

and linear constitutive equations:

$$\begin{aligned} t_{kl} &= A_{klmn}\varepsilon_{mn} + C_{klmn}\gamma_{mn}, \\ m_{kl} &= C_{mnlk}\varepsilon_{mn} + B_{lkmn}\gamma_{mn}. \end{aligned} \tag{A.10}$$

Here $l_k$, $\phi_k$ and $m_{kl}$ are the body couple, microrotation vector and the couple stress respectively and the constitutive coefficients are

$$\begin{aligned} A_{klmn} &= \lambda\delta_{kl}\delta_{mn} + \left(\mu+\frac{\chi}{2}\right)\delta_{km}\delta_{\ln} + \left(\mu-\frac{\chi}{2}\right)\delta_{kn}\delta_{lm}, \quad C_{klmn} = 0, \\ B_{klmn} &= \alpha\delta_{kl}\delta_{mn} + \beta\delta_{kn}\delta_{lm} + \gamma\delta_{km}\delta_{\ln}. \end{aligned} \tag{A.11}$$

Substituting Eqs. (A.11) and (A.9) into Eq. (A.10) and then into Eq. (A.8), we obtain

$$\begin{aligned} &\left(\mu+\frac{\chi}{2}\right)u_{l,kk} + \left(\lambda+\mu-\frac{\chi}{2}\right)u_{k,kl} + \chi\in_{lkm}\phi_{m,k} + f_l = 0, \\ &\gamma\phi_{l,kk} + (\alpha+\beta)\phi_{k,kl} - 2\chi\phi_l + \chi\in_{lkm}u_{m,k} + l_l = 0. \end{aligned} \tag{A.12}$$

Here $\alpha$, $\beta$, $\gamma$ and $\chi$ are new constitutive coefficients for the micropolar medium.

The results of the Galerkin's approach for micropolar medium are given by Sandru (1966) for two types of loadings. In the first case, the concentrated force $\mathbf{f} = \mathbf{q}\delta(x_1, x_2, x_3)$ is assumed acting at the origin of the coordinate system. The solutions for this case are

$$\begin{aligned} \mathbf{u} &= \frac{1}{16\pi\mu(1-v)}\left[(3-4v)\left(\frac{\mathbf{q}}{r}\right) + \left(\frac{(\mathbf{q}\cdot\mathbf{r})\mathbf{r}}{r^3}\right)\right] + \frac{\gamma}{16\pi\mu^2}\nabla\times\nabla\times\left[\frac{\mathbf{q}}{r}(\mathrm{e}^{-r/h_2}-1)\right], \\ \phi &= \frac{1}{8\pi\mu}\nabla\times\left[\frac{\mathbf{q}}{r}(1-\mathrm{e}^{-r/h_2})\right]. \end{aligned} \tag{A.13}$$

Here $v$ is Poisson's ratio.

For the second loading, $\mathbf{f}$ is assumed absent, and $\mathbf{l} = \mathbf{p}\delta(x_1, x_2, x_3)$ is acting at the origin. Then the solutions are

$$\begin{aligned} \mathbf{u} &= \frac{1}{8\pi\mu}\nabla \times \left[\frac{\mathbf{p}}{\mathrm{r}}(1 - \mathrm{e}^{-r/h_2})\right], \\ \phi &= \nabla\Lambda_0^* + \frac{2\mu+\chi}{16\pi\mu\chi}\nabla \times \nabla \times \left[\frac{\mathbf{p}}{r}\left(1 - \mathrm{e}^{-r/h_2}\right)\right]. \end{aligned} \tag{A.14}$$

Here

$$\Lambda_0^* = \frac{1}{8\pi\chi}\mathbf{p}\cdot\nabla\left[\frac{1}{r}(\mathrm{e}^{-r/h_3-1})\right]. \tag{A.15}$$

## Appendix B

### B.1. Eshelby tensors for microelongated media

$$\begin{aligned} \varepsilon_{kl}(\mathbf{x}) &= K_{klij}(\mathbf{x})\varepsilon_{ij}^* + L_{kl}(\mathbf{x})\phi^*, \\ \phi(\mathbf{x}) &= \hat{K}_{ij}(\mathbf{x})\varepsilon_{ij}^* + \hat{L}(\mathbf{x})\phi^*, \end{aligned} \tag{B.1}$$

$$\begin{aligned} K_{klij}(\mathbf{x}) &= I^{\mathrm{C}}_{lji,k}(\mathbf{x}) + I_{lji,k}(\mathbf{x}), \quad L_{kl}(\mathbf{x}) = J_{l,k}(\mathbf{x}), \\ \hat{K}_{ij}(\mathbf{x}) &= \hat{I}_{ji}(\mathbf{x}), \quad \hat{L}(\mathbf{x}) = \hat{J}(\mathbf{x}), \end{aligned} \tag{B.2}$$

$$\begin{aligned} I_{nji}(\mathbf{x}) &= -\frac{2(\lambda+\mu)\lambda_0^2}{(\lambda+2\mu)B_0}\delta_{ij}M_{1,n}(\mathbf{x}) + \frac{\lambda_0^2}{B_0}\delta_{ij}M_{3,n}(\mathbf{x},h_1) - \frac{\lambda_0^2}{(\lambda+2\mu)B_0}\mu M_{2,ijn}(\mathbf{x}) \\ &\quad - \frac{2a_0\lambda_0^2}{B_0^2}\mu M_{1,ijn}(\mathbf{x}) + \frac{a_0\lambda_0^2}{B_0^2}\lambda\delta_{ij}M_{3,kkn}(\mathbf{x},h_1) + \frac{2a_0\lambda_0^2}{B_0^2}\mu M_{3,ijn}(\mathbf{x},h_1), \\ J_n(\mathbf{x}) &= -2\frac{\lambda_0\lambda_1}{B_0}M_{1,n}(\mathbf{x}) + \frac{a_0\lambda_0\left[2\lambda_0^2 - \lambda_1(\lambda+2\mu)\right]}{B_0^2}M_{3,kkn}(\mathbf{x},h_1) + \frac{\lambda_1\lambda_0}{B_0}M_{3,n}(\mathbf{x},h_1), \\ \hat{I}_{ji}(\mathbf{x}) &= 2\frac{\lambda_0(\lambda+\mu)}{B_0}\delta_{ij} + \left[\frac{\lambda_0\mu}{B_0}M_{2,ijn}(\mathbf{x}) + \frac{2\lambda_0B_1\mu}{B_0^2}M_{1,ijn}(\mathbf{x}) - \frac{\lambda\lambda_0B_1}{B_0^2}\delta_{ij}M_{3,kkn}(\mathbf{x},h_1)\right. \\ &\quad \left. - \frac{2\lambda_0B_1\mu}{B_0^2}M_{3,ijn}(\mathbf{x},h_1) - \frac{\lambda_0(\lambda+2\mu)}{B_0}\delta_{ij}M_{3,n}(\mathbf{x},h_1)\right]\frac{1}{M_{1,n}(\mathbf{x})}, \\ \hat{J}(\mathbf{x}) &= \frac{\lambda_0^2 + \lambda_1(\lambda+\mu)}{B_0} - \left[\frac{B_1\left(2\lambda_0^2 - \lambda_1(\lambda+2\mu)\right)}{B_0^2}M_{3,kkn}(\mathbf{x},h_1) + \frac{\lambda_1(\lambda+2\mu)}{B_0}M_{3,n}(\mathbf{x},h_1)\right]\frac{1}{M_{1,n}(\mathbf{x})}, \\ I^{\mathrm{C}}_{nji}(\mathbf{x}) &= \frac{\lambda+\mu}{\lambda+2\mu}M_{2,ijn}(\mathbf{x}) - \frac{\lambda}{\lambda+2\mu}\delta_{ij}M_{1,n}(\mathbf{x}) - \delta_{in}M_{1,j}(\mathbf{x}) - \delta_{jn}M_{1,i}(\mathbf{x}). \end{aligned} \tag{B.3}$$

### B.2. Eshelby tensors for micropolar media

$$\begin{aligned} \varepsilon_{mn}(\mathbf{x}) &= K_{mnji}(\mathbf{x})\varepsilon_{ji}^* + L_{mnji}(\mathbf{x})\gamma_{ji}^*, \\ \gamma_{mn}(\mathbf{x}) &= \hat{K}_{mnji}(\mathbf{x})\varepsilon_{ji}^* + \hat{L}_{mnji}(\mathbf{x})\gamma_{ji}^*, \end{aligned} \tag{B.4}$$

$$K_{mnji}(\mathbf{x}) = I^{\mathrm{C}}_{nji,m}(\mathbf{x}) + I_{nji,m}(\mathbf{x}) - \in_{lmn}\hat{I}_{lji}(\mathbf{x}), \quad \hat{K}_{mnji}(\mathbf{x}) = \hat{I}_{nji,m}(\mathbf{x}),$$
$$L_{mnji}(\mathbf{x}) = J_{nji,m}(\mathbf{x}) - \in_{lmn}\hat{J}_{lji}(\mathbf{x}), \quad \hat{L}_{mnji}(\mathbf{x}) = \hat{J}_{nji,m}(\mathbf{x}), \tag{B.5}$$

$$\begin{aligned}
I_{nji}(\mathbf{x}) &= 2B\mu h_2^2 M_{1,ijn}(\mathbf{x}) + \frac{\chi}{\mu}\left[\delta_{jn}M_{1,i}(\mathbf{x}) - \delta_{in}M_{1,j}(\mathbf{x})\right] - Bh_2^2\left[\lambda\delta_{ij}M_{3,kkn}(\mathbf{x},h_2) + 2\mu M_{3,ijn}(\mathbf{x},h_2)\right] \\
&\quad + B\lambda\delta_{ij}M_{3,n}(\mathbf{x},h_2) + \left[B\left(\mu + \frac{\chi}{2}\right) + \frac{\chi}{2\mu}\right]\delta_{in}M_{3,j}(\mathbf{x},h_2) + \left[B\left(\mu - \frac{\chi}{2}\right) - \frac{\chi}{2\mu}\right]\delta_{jn}M_{3,i}(\mathbf{x},h_2), \\
J_{nji}(\mathbf{x}) &= -\frac{1}{2\mu}\left[\gamma\in_{nik}M_{1,jk}(\mathbf{x}) + \beta\in_{njk}M_{1,ik}(\mathbf{x})\right] + \frac{1}{2\mu}\left[\gamma\in_{nik}M_{3,jk}(\mathbf{x},h_2) + \beta\in_{njk}M_{3,ik}(\mathbf{x},h_2)\right], \\
\hat{I}_{nji}(\mathbf{x}) &= \frac{1}{4\mu}\left[\chi\in_{ijk}M_{1,kn}(\mathbf{x}) - (2\mu+\chi)\in_{nik}M_{1,jk}(\mathbf{x}) - (2\mu-\chi)\in_{njk}M_{1,ik}(\mathbf{x})\right] \\
&\quad - \frac{1}{4\mu}\left[(2\mu+\chi)\in_{ijk}M_{3,kn}(\mathbf{x},h_2) - (2\mu+\chi)\in_{nik}M_{3,jk}(\mathbf{x},h_2) - (2\mu-\chi)\in_{njk}M_{3,ik}(\mathbf{x},h_2)\right] \\
&\quad + \frac{1}{2}\in_{ijk}M_{3,kn}(\mathbf{x},h_3) + \frac{2\mu+\chi}{4\mu h_2^2}\in_{ijn}M_3(\mathbf{x},h_2), \\
\hat{J}_{nji}(\mathbf{x}) &= -\frac{\gamma+\beta}{4\mu}M_{1,ijn}(\mathbf{x}) + \frac{2\mu+\chi}{4\mu\chi}\left[\alpha\delta_{ij}M_{3,kkn}(\mathbf{x},h_2) + (\gamma+\beta)M_{3,ijn}(\mathbf{x},h_2)\right] \\
&\quad - \frac{1}{2\chi}\left[\alpha\delta_{ij}M_{3,kkn}(\mathbf{x},h_3) + (\gamma+\beta)M_{3,ijn}(\mathbf{x},h_3)\right] - \frac{2\mu+\chi}{4\mu\chi h_2^2}[\alpha\delta_{ij}M_{3,n}(\mathbf{x},h_2) + \gamma\delta_{in}M_{3,j}(\mathbf{x},h_2) \\
&\quad + \beta\delta_{jn}M_{3,i}(\mathbf{x},h_2)], \\
I^{\mathrm{C}}_{nji}(\mathbf{x}) &= \frac{\lambda+\mu}{\lambda+2\mu}M_{2,ijn}(\mathbf{x}) - \frac{\lambda}{\lambda+2\mu}\delta_{ij}M_{1,n}(\mathbf{x}) - \delta_{in}M_{1,j}(\mathbf{x}) - \delta_{jn}M_{1,i}(\mathbf{x}).
\end{aligned} \tag{B.6}$$

## References


Benveniste, Y., 1987. Mech. Mater. 6, 147–157.
Cheng, Z.Q., He, L.H., 1995. Int. J. Eng. Sci. 33, 389–397.
Cheng, Z.Q., He, L.H., 1997. Int. J. Eng. Sci. 35, 659–668.
Eringen, A.C., 1966. J. Math. Mech. 15, 909–923.
Eringen, A.C., 1990. Int. J. Eng. Sci. 28, 1291–1301.
Eringen, A.C., 1999. Microcontinuum Field Theories: Foundations and Solids. Springer-Verlag, New York.
Eringen, A.C., Suhubi, E.S., 1964. Int. J. Eng. Sci. 2, 189–203.
Eshelby, J.D., 1957. Proc. Roy. Soc. Lond. A 241, 376–396.
Galerkin, B., 1930. C.R. Acad. Sci. 190, 1047–1048.
Hsieh, R.K.T., 1982. Int. J. Eng. Sci. 20, 261–270.
Hsieh, R.K.T., Vörös, G., Kovács, I., 1980. Physica B 101, 201–208.
Inan, E., 1990. Elastic waves in damaged material. in: Proc. of Fourth ICOVP-99, A, Calcutta, India, pp. 149–160.
Kiris, A., Inan, E., 2005. Int. J. Eng. Sci. 43, 49–58.
Mori, T., Tanaka, K., 1973. Acta Metall. 21, 571–574.
Mura, T., 1982. Micromechanics of Defects in Solids. Kluwer, The Netherlands.
Sandru, N., 1966. Int. J. Eng. Sci. 4, 81–94.
Sharma, P., Dasgupta, A., 2002. Phys. Rev. B 66, 224110-1–224110-10.
Suhubi, E.S., Eringen, A.C., 1964. Int. J. Eng. Sci. 2, 389–404.